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Author(s)	SUZUKI, MASA AKI
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AN EPIMORPHISM BETWEEN KNOT GROUPS WHICH DOES NOT MAP A MERIDIAN TO A MERIDIAN (II)

MASAAKI SUZUKI

1. INTRODUCTION

Let K be a knot in S^3 and $G(K)$ the knot group. The existence of an epimorphism between knot groups defines a partial order on the set of prime knots. In [4], we consider epimorphisms which map meridians to a meridians. it is determined whether there exists such an epimorphism between the knot groups for each pair of prime knots with up to 10 crossings in [4]. The result of [4] is extended to prime knots with up to 11 crossings in [2]. On the other hand, we show an example of an epimorphism which does not map a meridian to a meridian in [9]. In this paper, we will show another expmple of an epimorphism, whose image is the knot group of the figure eight knot.

2. DEFINITION OF AN EPIMORPHISM AND MAIN THEOREM

Let K_1 be knots as depicted in Figure 1 and K_2 the figure eight knot.

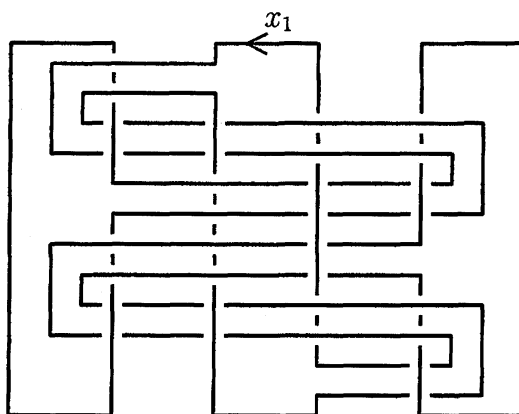


FIGURE 1. Knot K_1

The knot group $G(K_1)$ admits a Wirtinger presentation with respect to Figure 1. We denote the generators by x_1, x_2, \dots, x_{32} and the defining relators are

$$\begin{array}{llllll} x_5 x_1 \bar{x}_5 \bar{x}_2, & x_{13} x_3 \bar{x}_{13} \bar{x}_2, & x_{25} x_3 \bar{x}_{25} \bar{x}_4, & x_{31} x_5 \bar{x}_{31} \bar{x}_4, & x_{13} x_6 \bar{x}_{13} \bar{x}_5, & x_1 x_6 \bar{x}_1 \bar{x}_7, \\ x_{20} x_8 \bar{x}_{20} \bar{x}_7, & x_{26} x_8 \bar{x}_{26} \bar{x}_9, & x_{31} x_9 \bar{x}_{31} \bar{x}_{10}, & x_{25} x_{11} \bar{x}_{25} \bar{x}_{10}, & x_{13} x_{11} \bar{x}_{13} \bar{x}_{12}, & x_5 x_{13} \bar{x}_5 \bar{x}_{12}, \\ x_5 x_{14} \bar{x}_5 \bar{x}_{13}, & x_9 x_{14} \bar{x}_9 \bar{x}_{15}, & x_{26} x_{16} \bar{x}_{26} \bar{x}_{15}, & x_{20} x_{16} \bar{x}_{20} \bar{x}_{17}, & x_{23} x_{18} \bar{x}_{23} \bar{x}_{17}, & x_{17} x_{18} \bar{x}_{17} \bar{x}_{19}, \\ x_7 x_{20} \bar{x}_7 \bar{x}_{19}, & x_{31} x_{20} \bar{x}_{31} \bar{x}_{21}, & x_{18} x_{22} \bar{x}_{18} \bar{x}_{21}, & x_{28} x_{22} \bar{x}_{28} \bar{x}_{23}, & x_{11} x_{24} \bar{x}_{11} \bar{x}_{23}, & x_3 x_{24} \bar{x}_3 \bar{x}_{25}, \\ x_{31} x_{26} \bar{x}_{31} \bar{x}_{25}, & x_7 x_{26} \bar{x}_7 \bar{x}_{27}, & x_{17} x_{28} \bar{x}_{17} \bar{x}_{27}, & x_{23} x_{28} \bar{x}_{23} \bar{x}_{29}, & x_{28} x_{30} \bar{x}_{28} \bar{x}_{29}, & x_{18} x_{30} \bar{x}_{18} \bar{x}_{31}, \\ x_3 x_{32} \bar{x}_3 \bar{x}_{31}, & x_{11} x_{32} \bar{x}_{11} \bar{x}_1, & & & & \end{array}$$

where $\bar{x}_i = x_i^{-1}$. Note that x_1 can be regarded as a meridian of K_1 since all the generators are conjugate to one another.

We fix a presentation of the knot group $G(K_2)$:

$$G(K_2) = \langle y_1, y_2 \mid \bar{y}_1 y_2 y_1 \bar{y}_2 y_1 y_2 \bar{y}_1 \bar{y}_2 y_1 \bar{y}_2 \rangle.$$

where $\bar{y}_i = y_i^{-1}$ again.

Let $f : G(K_1) \rightarrow G(K_2)$ be a mapping defined by the image of generators of $G(K_1)$ as follows. Here we write numbers 1, 2 for the generators y_1, y_2 respectively. For example, $11\bar{2}$ means $y_1 y_1 y_2^{-1}$.

$$\begin{array}{ll} f(x_1) = 11\bar{2}, & f(x_2) = 1\bar{2}111\bar{2}\bar{1}\bar{2}\bar{1}, \\ f(x_3) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, & f(x_4) = 11\bar{2}1\bar{2}12\bar{1}\bar{1}, \\ f(x_5) = 1\bar{2}1, & f(x_6) = 2\bar{1}\bar{2}111\bar{2}, \\ f(x_7) = 1\bar{2}1, & f(x_8) = 2\bar{1}\bar{2}\bar{1}\bar{2}1\bar{2}121\bar{2}, \\ f(x_9) = 1\bar{2}1, & f(x_{10}) = 11\bar{2}1\bar{2}12\bar{1}\bar{1}, \\ f(x_{11}) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, & f(x_{12}) = 1\bar{2}111\bar{2}\bar{1}\bar{2}\bar{1}, \\ f(x_{13}) = 11\bar{2}, & f(x_{14}) = 1\bar{2}1\bar{2}1\bar{2}1, \\ f(x_{15}) = 11\bar{2}, & f(x_{16}) = 2\bar{1}\bar{2}\bar{1}\bar{2}111\bar{2}\bar{1}\bar{2}, \\ f(x_{17}) = 11\bar{2}, & f(x_{18}) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, \\ f(x_{19}) = 1\bar{2}121\bar{2}1\bar{2}\bar{1}\bar{2}\bar{1}, & f(x_{20}) = 21\bar{2}1\bar{2}, \\ f(x_{21}) = 111\bar{2}\bar{1}, & f(x_{22}) = 12\bar{1}\bar{1}\bar{1}\bar{2}1\bar{2}1\bar{2}111\bar{2}\bar{1}, \\ f(x_{23}) = 111\bar{2}\bar{1}, & f(x_{24}) = 12\bar{1}\bar{1}\bar{1}\bar{2}1\bar{2}1\bar{2}111\bar{2}\bar{1}, \\ f(x_{25}) = 111\bar{2}\bar{1}, & f(x_{26}) = 21\bar{2}1\bar{2}, \\ f(x_{27}) = 1\bar{2}121\bar{2}1\bar{2}\bar{1}\bar{2}\bar{1}, & f(x_{28}) = 2\bar{1}\bar{2}1121\bar{2}\bar{1}, \\ f(x_{29}) = 11\bar{2}, & f(x_{30}) = 12\bar{1}\bar{1}\bar{2}\bar{1}\bar{2}1\bar{2}1\bar{2}111\bar{2}\bar{1}, \\ f(x_{31}) = 11\bar{2}, & f(x_{32}) = 12\bar{1}\bar{1}\bar{2}\bar{1}\bar{2}1\bar{2}1\bar{2}111\bar{2}\bar{1}. \end{array}$$

Theorem 2.1. *The above mapping $f : G(K_1) \rightarrow G(K_2)$ is an epimorphism which does not map a meridian of K_1 to a meridian of K_2 .*

3. PROOF

Theorem 2.1 is shown in this section. First, we will verify the defining relators of $G(K_1)$ vanish under the mapping f so that we prove that $f : G(K_1) \rightarrow G(K_2)$ is a group homomorphism,

$$\begin{aligned} f(x_5 x_1 \bar{x}_5 \bar{x}_2) &= 1\bar{2}1 \cdot 11\bar{2} \cdot 1\bar{2}\bar{1} \cdot 1\bar{2}12\bar{1}\bar{1}\bar{1}\bar{2}\bar{1} = e, \\ f(x_{13} x_3 \bar{x}_{13} \bar{x}_2) &= 11\bar{2} \cdot 2\bar{1}\bar{2}1121\bar{2}\bar{1} \cdot 2\bar{1}\bar{1} \cdot 1\bar{2}12\bar{1}\bar{1}\bar{1}\bar{2}\bar{1} = e, \\ f(x_{25} x_3 \bar{x}_{25} \bar{x}_4) &= 111\bar{2}\bar{1} \cdot 2\bar{1}\bar{2}1121\bar{2}\bar{1} \cdot 12\bar{1}\bar{1}\bar{1} \cdot 11\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}\bar{1} \\ &= 111\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}1\bar{2}\bar{1}\bar{2}\bar{1}\bar{1} = e, \\ f(x_{31} x_5 \bar{x}_{31} \bar{x}_4) &= 11\bar{2} \cdot 1\bar{2}1 \cdot 2\bar{1}\bar{1} \cdot 11\bar{2}\bar{1}\bar{2}\bar{1}\bar{2}\bar{1}\bar{1} = e, \\ f(x_{13} x_6 \bar{x}_{13} \bar{x}_5) &= 11\bar{2} \cdot 2\bar{1}\bar{2}111\bar{2} \cdot 2\bar{1}\bar{1} \cdot 1\bar{2}\bar{1} = e, \\ f(x_1 x_6 \bar{x}_1 \bar{x}_7) &= 11\bar{2} \cdot 2\bar{1}\bar{2}111\bar{2} \cdot 2\bar{1}\bar{1} \cdot 1\bar{2}\bar{1} = e. \end{aligned}$$

Then $f : G(K_1) \rightarrow G(K_2)$ is a group homomorphism by the above and similar calculations. Next, we will show that the group homomorphism $f : G(K_1) \rightarrow G(K_2)$ is surjective. We

consider two elements $x_1\bar{x}_7x_1\bar{x}_{23}x_1, \bar{x}_7x_1\bar{x}_{23}x_1x_1\bar{x}_7x_1\bar{x}_{23}x_1$ of $G(K_1)$.

$$\begin{aligned}
 f(x_1\bar{x}_7x_1\bar{x}_{23}x_1) &= f(x_1) \overline{f(x_7)} f(x_1) \overline{f(x_{23})} f(x_1) \\
 &= 11\bar{2} \cdot \bar{1}2\bar{1} \cdot 11\bar{2} \cdot 12\bar{1}\bar{1}\bar{1} \cdot 11\bar{2} = 11\bar{2}\bar{1}21\bar{2}12\bar{1}\bar{2} = 1, \\
 f(\bar{x}_7x_1\bar{x}_{23}x_1x_1\bar{x}_7x_1\bar{x}_{23}x_1) \\
 &= \overline{f(x_7)} f(x_1) \overline{f(x_{23})} f(x_1) f(x_1) \overline{f(x_7)} f(x_1) \overline{f(x_{23})} f(x_1) \\
 &= \bar{1}2\bar{1} \cdot 11\bar{2} \cdot 12\bar{1}\bar{1}\bar{1} \cdot 11\bar{2} \cdot 11\bar{2} \cdot \bar{1}2\bar{1} \cdot 11\bar{2} \cdot 12\bar{1}\bar{1}\bar{1} \cdot 11\bar{2} \\
 &= \bar{1}21\bar{2}12\bar{1}\bar{2}11\bar{2}\bar{1}21\bar{2}12\bar{1}\bar{2} = 2.
 \end{aligned}$$

Since two elements 1 and 2 generate $G(K_2)$, it is shown that the group homomorphism f is surjective. Finally, we will prove that f does not map a meridian of K_1 to a meridian of K_2 . We can fix meridians for K_1 and K_2 by x_1 and 1, without loss of generality. Let $\rho : G(K_2) \rightarrow SL(2; \mathbb{Z}/3\mathbb{Z})$ be a representation of $G(K_2)$ defined by

$$\rho(1) = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad \rho(2) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

We can check easily that ρ is a representation of $G(K_2)$. Besides, we get

$$\rho(f(x_1)) = \rho(11\bar{2}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Note that the trace of $\rho(1)$ is 1. On the other hand, the trace of $\rho(f(x_1))$ is not equal to 1. Hence $f(x_1)$ is not conjugate to 1. It follows that the epimorphism f does not map a meridian of K_1 to a meridian of K_2 . This completes the proof.

4. PROBLEM

In this section, we propose some problems related to epimorphisms between knot groups. We determined whether there exist an epimorphism mapping a meridian to a meridian between the knot groups of each pair of prime knots with up to 11 crossings in [4] and [2].

Problem 4.1. *Determine whether there exist an epimorphism between the knot groups of each pair of prime knots with up to 11 crossings. Here we do not assume that an epimorphism maps a meridian to a meridian. In particular, does there exist such an epimorphism between 2-bridge knot groups?*

We note that Ohtsuki-Riley-Sakuma [7] and Lee-Sakuma [6] studied epimorphisms between 2-bridge link groups. Perhaps there exist several epimorphisms for the given knot groups. In this sense, we are interested in the following.

Problem 4.2. *Which pair of knots with up to 11 crossings admit an epimorphism between their knot groups which does not map a meridian to a meridian?*

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DEPARTMENT OF MATHEMATICS, AKITA UNIVERSITY

E-mail address: macky@math.akita-u.ac.jp